

# SUPER-IDEALS IN BANACH SPACES

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**ABSTRACT.** A natural class of ideals, super-ideals, of Banach spaces is defined and studied. The motivation for working with this class of subspaces is our observations that they inherit diameter 2 properties and the Daugavet property. Lindenstrauss spaces are known to be the class of Banach spaces which are ideals in every superspace; we show that being a super-ideal in every superspace characterizes the class of Gurarii spaces.

## 1. INTRODUCTION AND MOTIVATION

Let  $Y$  be a (real) Banach space and  $X$  a subspace (not necessarily closed). Recall that  $X$  is called an *ideal* in  $Y$  if for every  $\varepsilon > 0$  and every finite-dimensional subspace  $E \subset Y$  there exists a linear  $T : E \rightarrow X$  such that

- (i)  $Te = e$  for all  $e \in X \cap E$ ,
- (ii)  $\|Te\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ .

Using a compactness argument of Lindenstrauss, it can be seen that (i) and (ii) above imply the existence of a Hahn-Banach extension operator  $\varphi : X^* \rightarrow Y^*$ . By existence of a Hahn-Banach extension operator, we mean that the norm-preserving extensions, guaranteed by the Hahn-Banach theorem, can be chosen in a linear fashion. Moreover,  $\varphi$  and  $T$  are strongly related; a proof of the following theorem can be found in [OP07]:

**Theorem 1.1.** *Assume that  $X$  is an ideal in  $Y$ . Then there exists a Hahn-Banach extension operator  $\varphi : X^* \rightarrow Y^*$  such that for every  $\varepsilon > 0$ , every finite-dimensional subspace  $E \subset Y$  and every finite-dimensional subspace  $F \subset X^*$  there exists  $T : E \rightarrow X$  such that*

- (i)  $Te = e$  for all  $e \in X \cap E$ ,
- (ii)  $\|Te\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ , and
- (iii)  $\varphi f^*(e) = f^*(Te)$  for all  $e \in E, f^* \in F$ .

Using the Principle of Local Reflexivity it is quick to show that when  $X$  is closed, the existence of a Hahn-Banach extension operator implies that  $X$  is an ideal.

The connection between ideals and subspaces for which there exists a Hahn-Banach extension operator dates back to a 1972-paper of Fakhoury [Fak72]. In that paper this property is referred to as the pair  $(X, Y)$  being *admissible*. Later on the expression  $X$  is *locally 1-complemented* is often used; this terminology seems to date back to the paper [Kal84] of Kalton. After the paper [GKS93] the term ideal seems to be most used, and we have decided to use this term as well.

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Let us now describe the content of our paper: In searching for a natural condition assuring  $X$  to inherit the property from its superspace  $Y$  that every non-void relatively weakly open subset of  $B_Y$  has diameter 2, we observed that if the  $T$  in Theorem 1.1 can be assumed to be an  $\varepsilon$ -isometry, then this *diameter 2 property* passes down to  $X$  from  $Y$ . This observation is presented in Proposition 3.2. Also, we observed that the same condition works for the problem of inheriting the Daugavet property. The presentation of this result can be found in Proposition 3.8. Precise definitions and necessary background on both diameter 2 properties and the Daugavet property are incorporated in the presentation in Section 3.

Before going further, note that  $X$  just being an ideal is certainly not enough to inherit neither the diameter 2 property nor the Daugavet property. In fact, every Banach space can be realized as a 1-complemented subspace of a Banach space with both the diameter 2 property and the Daugavet property, see Proposition 3.9.

The above results on inheriting the diameter 2 property and the Daugavet property indicate that subspaces obeying the conclusion in Theorem 1.1 with  $T$  almost isometric are of some relevance, and so we think it is natural to find out what can be said in general about such subspaces:

**Definition 1.2.** Let  $Y$  be a Banach space and  $X$  a subspace (not necessarily closed).  $X$  is called a *super-ideal* in  $Y$  if for every  $\varepsilon > 0$  and every finite-dimensional subspace  $E \subset Y$  there exists  $T : E \rightarrow X$  such that

- (i)  $Te = e$  for all  $e \in X \cap E$ .
- (ii)  $(1 + \varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ .

The immediate question is whether the analogue of Theorem 1.1 holds. Lindenstrauss' compactness argument of course produces a Hahn-Banach extension operator, but the problem is that one might risk to loose the  $\varepsilon$ -isometry property of  $T$ . It turns out that the analogue of Theorem 1.1 is true:

**Theorem 1.3.** Assume that  $X$  is a super-ideal in  $Y$ . Then there exists a Hahn-Banach operator  $\varphi : X^* \rightarrow Y^*$  such that for every  $\varepsilon > 0$ , every finite-dimensional subspace  $E \subset Y$  and every finite-dimensional subspace  $F \subset X^*$  there exists  $T : E \rightarrow X$  such that

- (i)  $Te = e$  for all  $e \in X \cap E$ ,
- (ii)  $(1 + \varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ , and
- (iii)  $\varphi f^*(e) = f^*(Te)$  for all  $e \in E, f^* \in F$ .

The proof of this structure result will be the starting point of Section 2.

Note that the conclusion of Theorem 1.3 is very similar to the Principle of Local Reflexivity, so a Banach space  $X$  is always a super-ideal in its bidual  $X^{**}$ . By Goldstine's theorem, in the Principle of Local Reflexivity setting, the range of  $\varphi : X^* \rightarrow X^{***}$  is 1-norming for  $X^{**}$  (in this case  $\varphi$  is simply the canonical embedding of  $X^*$  into  $X^{***}$ ). We will see in Proposition 2.2 that it is in general true that when the range of  $\varphi$  in Theorem 1.1 is 1-norming for  $Y$ , then the ideal  $X$  is a super-ideal in  $Y$ .

Knowing this, our next question is naturally whether for a super-ideal the associated Hahn-Banach extension operators  $\varphi$  from  $X^*$  into  $Y^*$  must have

range which is 1-norming for  $Y$ . We will see that this is not so in general; in Example 1 we will see that the 1-co-dimensional subspace  $X = \{(a_n)_{n=1}^\infty \in c_0 : a_1 = 0\}$  of  $c_0$  is a counterexample.

However, in the very important case of u-ideals the  $\varepsilon$ -isometry condition of T and having 1-norming range are indeed equivalent (Theorem 2.4). U-ideals were introduced and studied in [GKS93]; we also give some necessary background on u-ideals in the introduction to our Theorem 2.4.

To sum up, our motivation is the passing of diameter 2 properties and the Daugavet property to subspaces. This leads to the concept of a super-ideal. Super-ideals are studied in Section 2, while the results on the diameter 2 properties and the Daugavet property form Section 3.

In Section 4 we characterize Gurarii spaces in terms of super-ideals: From [Rao01] it is known that the Banach spaces which is an ideal in every superspace is exactly the class of Lindenstrauss spaces. We observe in Theorem 4.3 that the class of spaces which is a super-ideal in every superspace is the Gurarii spaces. From this it follows that Gurarii spaces have the Daugavet property. We end the paper by proving that Lindenstrauss spaces in general enjoy the diameter 2 properties.

We use standard Banach space notation; symbols and terms will however be carefully explained throughout the text when we think it is helpful to the reader. The reader only interested in the results on the passage of diameter 2 properties or the Daugavet property to subspaces may go directly to Section 3.

## 2. SUPER-IDEALS

We start by proving our main structure theorem:

**Theorem 2.1.** *Let  $Y$  be a Banach space and  $X$  a subspace (not necessarily closed). If  $X$  is a super-ideal in  $Y$ , then there exists a Hahn-Banach operator  $\varphi : X^* \rightarrow Y^*$  such that for every  $\varepsilon > 0$ , every finite-dimensional subspace  $E \subset Y$  and every finite-dimensional subspace  $F \subset X^*$  there exists  $T : E \rightarrow X$  such that*

- (i)  $Te = e$  for all  $e \in X \cap E$ ,
- (ii)  $(1 + \varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ , and
- (iii)  $\varphi f^*(e) = f^*(Te)$  for all  $e \in E, f^* \in F$ .

*Proof.* We first construct  $\varphi$  using a Lindenstrauss compactness argument. Order the set  $A = \{(E, F, \varepsilon)\}$ , where  $E \subset Y$  and  $F \subset X^*$  are finite-dimensional and  $\varepsilon > 0$ , by  $(E_1, F_1, \varepsilon_1) \leq (E_2, F_2, \varepsilon_2)$  if  $E_1 \subset E_2$ ,  $F_1 \subset F_2$ , and  $\varepsilon_2 \leq \varepsilon_1$ .

For  $\alpha \in A$ ,  $\alpha = (E, F, \varepsilon)$ , choose  $T_\alpha : E \rightarrow X$  satisfying (i) and (ii) in Definition 1.2. Define  $L_\alpha : Y \rightarrow X^{**}$  by  $L_\alpha y = T_\alpha y$  if  $y \in E$  and  $L_\alpha y = 0$  if  $y \notin E$ . We consider  $(L_\alpha) \subset \Pi_{y \in Y} B_{X^{**}}(0, 2\|y\|)$  which by Tychonoff is compact in the product weak\* topology. Without loss of generality we assume  $(L_\alpha)$  is convergent to some  $S \in \Pi_{y \in Y} B_{X^{**}}(0, 2\|y\|)$ . Note that this implies that for every finite number of elements  $(y_i)_{i=1}^n$  in  $Y$  and  $(x_j^*)_{j=1}^m$  in  $X^*$  we have

$$(2.1) \quad x_j^*(L_\alpha y_i) \rightarrow x_j^*(S y_i).$$

By construction  $Sy = y$  for every  $y \in X$ . It is also clear that  $\|S\| = 1$ , so that  $\varphi = S^*|_{X^*} : X^* \rightarrow Y^*$  is a Hahn-Banach extension operator.

Next we apply a perturbation argument due to Johnson, Rosenthal, and Zippin [JRZ71] as in [OP07].

Let  $(x_i^*, x_i)_{i=1}^n$  be a complete biorthogonal system for  $F$ . Define

$$Q = \sum_{i=1}^n i_X x_i \otimes \varphi(x_i^*).$$

Here  $i_X : X \rightarrow Y$  is the identity embedding. Then  $Q \in \mathcal{F}(Y^*, Y^*)$  is a projection with  $Q(Y^*) = \varphi(F)$ , and  $Q^*(Y^{**}) \subset X$ . Similarly let  $P \in \mathcal{F}(E, E)$  be a projection with  $P(E) = E \cap X$ .

For  $\alpha \in A$ ,  $\alpha = (E, F, \varepsilon)$ , let  $(T_\alpha)$  be the net from the first paragraph. Define  $S_\alpha : E \rightarrow X$  by

$$\begin{aligned} S_\alpha &= i_E P + T_\alpha (I_E - P) - Q^*(T_\alpha - i_E)(I_E - P) \\ &= i_E + (I_{Y^{**}} - Q^*)(T_\alpha - i_E)(I_E - P). \end{aligned}$$

Here  $i_E : E \rightarrow Y$  denotes the identity embedding. Now  $S_\alpha \in \mathcal{F}(E, X)$ , because  $i_E P(E) = E \cap X \subset X$  and  $Q^*(Y^{**}) \subset X$  and  $P, T_\alpha$ , and  $i_E$  are finite-rank operators. We have  $S_\alpha e = e$  for every  $e \in E \cap X$  because  $E \cap X = P(E)$  and  $P$  is a projection.

Let  $f^* \in F$  and  $e \in E$ . Using  $S_\alpha(E) \subset X$  we have

$$\begin{aligned} \langle f^*, S_\alpha e \rangle &= \langle \varphi f^*, S_\alpha e \rangle \\ &= \langle \varphi f^*, i_E e \rangle + \langle \varphi f^*, (I_{Y^{**}} - Q^*)(T_\alpha - i_E)(I_E - P)e \rangle \\ &= \langle \varphi f^*, i_E e \rangle + \langle (I_{Y^*} - Q)\varphi f^*, (T_\alpha - i_E)(I_E - P)e \rangle \\ &= \langle \varphi f^*, i_E e \rangle \end{aligned}$$

since  $Q(\varphi f^*) = \varphi f^*$ .

So far we have shown that  $(S_\alpha)$  satisfies (i) and (iii). Far out in the net the  $S_\alpha$ 's will inherit (ii) from the  $T_\alpha$ 's if we can show that  $\|S_\alpha - T_\alpha\|$  can be made as small as we wish. Note that

$$S_\alpha - T_\alpha = (i_E - T_\alpha)P - Q^*(T_\alpha - i_E)(I_E - P) = -Q^*(T_\alpha - i_E)(I_E - P)$$

since  $T_\alpha e = e$  for all  $e \in P(E)$ . Thus we have

$$\|S_\alpha - T_\alpha\| = \sup_{\|e\|=1} \|Q^*(T_\alpha - i_E)e\| \leq \sup_{\|e\|=1} \sum_{i=1}^n \|x_i\| |x_i^*(T_\alpha e) - \varphi x_i^*(e)|.$$

Let  $\alpha = (E, F, \varepsilon)$ . Let  $\delta > 0$  and choose a  $\delta$ -net  $(e_j)_{j=1}^k$  for  $S_E$ . We choose  $\beta \geq \alpha$  so that  $|x_i^*(T_\beta e_j) - \varphi x_i^*(e_j)| < \delta$  for every  $i = 1, \dots, n$  and  $j = 1, \dots, k$  using (2.1).

For  $e \in S_E$  choose  $j$  such that  $\|e - e_j\| < \delta$ , then

$$\begin{aligned} |x_i^*(T_\beta e) - \varphi x_i^*(e)| &\leq |x_i^*(T_\beta e) - x_i^*(T_\beta e_j)| + |x_i^*(T_\beta e_j) - \varphi x_i^*(e_j)| \\ &\quad + |\varphi x_i^*(e_j) - \varphi x_i^*(e)| \\ &\leq 2\|x_i^*\|\delta + \delta + \|x_i^*\|\delta \leq \delta(1 + 3\max_i \|x_i^*\|). \end{aligned}$$

By choosing  $\delta$  small enough we get that  $S_\beta$  satisfies (i), (ii) and (iii) for the given  $\alpha = (E, F, \varepsilon)$ . The desired  $T : E \rightarrow X$  is then  $S_\beta$ .  $\square$

As seen in the proof above, the Hahn-Banach extension operator  $\varphi$  stems from a norm one extension  $S : Y \rightarrow X^{**}$  of the canonical embedding  $k_X : X \rightarrow X^{**}$ , by  $\varphi = S^*|_{X^*}$ . Clearly, the existence of a Hahn-Banach extension operator and a norm-preserving extension of  $k_X$  to  $Y$  are equivalent. Moreover, the existence of a Hahn-Banach extension operator  $\varphi : X^* \rightarrow Y^*$  is equivalent to the existence of a norm one projection  $P$  on  $Y^*$  with  $\ker P = X^\perp$  and range equal to  $\varphi(X^*)$ .

From the way  $P, \varphi$  and  $S$  are connected, one obtains that the range of  $\varphi$  (or  $P$ ) is 1-norming if and only if  $S$  is an isometry into. This situation is well-studied in the recent literature (see e.g. [Rao01] or [LL09]); these ideals are called *strict ideals*.

**Proposition 2.2.** *Suppose  $X$  is a strict ideal in  $Y$ . Then  $X$  is a super-ideal in  $Y$ .*

*Proof.* The following argument is well-known (see e.g. [OP07, Lemma 2.2]). Suppose, for  $\varepsilon > 0$  and finite-dimensional  $E \subset Y$ ,  $F \subset X^*$ ,  $T$  has been found such that (i), (ii) and (iii) in Theorem 1.1 holds with  $\varphi(X^*)$  1-norming for  $Y$ . Pick  $\delta > 0$  such that  $(1 - 2\delta)^{-1} < 1 + \varepsilon$ , choose a  $\delta$ -net  $(e_i)_{i=1}^n$  for  $S_E$  and, thereafter, find  $(y_i^*)_{i=1}^n \subset S_{Y^*}$  such that  $1 - \delta < y_i^*(e_i)$ . Now, if  $e \in S_E$  is arbitrary, clearly  $1 - 2\delta < |y_i^*(e)|$  for some  $y_i^*$ .

Since  $\varphi(X^*)$  is 1-norming for  $Y$ , we may even assume  $y_i^* = \varphi x_i^*$  for  $x_i^* \in S_{X^*}$ , and this is what makes things work, because now, if  $e \in S_E$ ,

$$1 - 2\delta < |\varphi x_i^*(e)| = |x_i^*(Te)| \leq \|Te\|.$$

Thus  $\|Te\| > 1 - 2\delta > (1 + \varepsilon)^{-1}\|e\|$ , and the proof is completed by a homogeneity argument.  $\square$

*Remark 2.1.* Note that for *closed* strict ideals Proposition 2.2 also follows by combining the isometry  $S$  with a suitable use of the Principle of Local Reflexivity.

We now give an example which shows that the converse of Proposition 2.2 is not true. For this example we will just need a little more background on ideals. An ideal  $X$  in  $Y$  is said to have the *unique ideal property* if there is only one possible extension operator  $\varphi$ . This is a rather special situation, but it still occurs in some cases. A closed ideal  $X \subset Y$  is an M-ideal in  $Y$  if the ideal projection  $P : Y^* \rightarrow Y^*$  is an L-projection, that is,

$$\|y^*\| = \|Py^*\| + \|y^* - Py^*\| \quad \text{for all } y^* \in Y^*.$$

A particular case of this situation is when  $X$  is 1-complemented in  $Y$  by an M-projection  $Q$ , that is  $QY = X$  and

$$\|y\| = \max\{\|Qy\|, \|y - Qy\|\} \quad \text{for all } y \in Y,$$

in this case  $X$  is called an M-summand in  $Y$ . By [HWW93, Proposition I.1.12] M-ideals have the unique ideal property. Further, if  $X$  is also an M-summand, then  $\varphi(X^*)$  is weak\* closed, hence if  $X$  is a proper subspace of  $Y$  it can't be a strict ideal in  $Y$ .

We denote by  $e_n$  the  $n$ -th standard basis vector in  $c_0$  and by  $e_n^*$  the biorthonormal vector in  $\ell_1$ .

EXAMPLE 1. *The subspace  $X = \{(a_n)_{n=1}^\infty \in c_0 : a_1 = 0\} = \ker e_1^*$  of  $c_0$  is 1-complemented and a super-ideal in  $c_0$ .*

*Proof.* Clearly  $X$  is a proper M-summand in  $c_0$  by the projection  $Q$  putting 0 on the first coordinate, so, by the above remarks, what is left to show is that  $X$  is a super-ideal.

Let  $E$  be a finite-dimensional subspace of  $c_0$  and let  $(x_i)_{i=1}^m$  be some  $\varepsilon$ -net for  $S_E$ . Find  $N$  such that  $|x_i(N)| < \varepsilon$  for  $i = 1, 2, \dots, m$ . Define  $T : E \rightarrow X$  by  $T(y) = Qy + e_1^*(y)e_N$ . Then  $T$  is obviously linear and an  $\varepsilon$ -isometry on  $(x_i)_{i=1}^m$ . By [AK06, Lemma 11.1.11]  $T$  is an almost-isometry on all of  $E$ .  $\square$

As we have seen from Example 1 super-ideals need not be strict. We will now show that if some symmetry condition is imposed, then super-ideals indeed are strict. The closed subspace  $X$  is said to be a u-ideal in  $Y$  if there exists an ideal projection  $P : Y^* \rightarrow Y^*$  such that  $\|I - 2P\| = 1$  ( $P$  is *unconditional*). If the range of  $P$  is 1-norming for  $Y$ ,  $X$  is called a strict u-ideal in  $Y$ . There is never more than one unconditional  $P$  ([GKS93, Lemma 3.1]). Further, every M-ideal is a u-ideal. From [GKS93, Proposition 3.6] it is known that  $X$  is a u-ideal if and only if  $X$  is an ideal with the extra condition

$$(iv) \|e - 2T(e)\| \leq (1 + \varepsilon)\|e\| \text{ for all } e \in E.$$

We will now assume that  $X$  is a u-ideal in  $Y$  and that the  $T$ 's above can be chosen to be almost isometries:

**Definition 2.3.** A closed subspace  $X$  is called a super u-ideal in  $Y$  if for every  $\varepsilon > 0$  and every finite-dimensional subspace  $E \subset Y$  there exists  $T : E \rightarrow X$  such that

- (i)  $Te = e$  for all  $e \in X \cap E$ ,
- (ii)  $(1 + \varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ , and
- (iv)  $\|e - 2T(e)\| \leq (1 + \varepsilon)\|e\|$  for all  $e \in E$ .

*Remark 2.2.* An inspection of the proof of Theorem 2.1 shows that when  $X$  is a super u-ideal in  $Y$ , it is possible to obtain a Hahn-Banach extension operator  $\varphi : X^* \rightarrow Y^*$  such that also

$$(iii) \varphi f^*(e) = f^*(Te) \text{ for all } e \in E, f^* \in F$$

is fulfilled. This observation, however, will not be needed for our purposes.

**Theorem 2.4.** *If  $X$  is a super u-ideal in  $Y$ , then  $X$  is a strict u-ideal in  $Y$ .*

*Proof.* Assume that  $X$  is a super u-ideal in  $Y$ . Choose  $y \in Y \setminus X$ . Then  $X$  is a super u-ideal in  $Z = \text{span}(X, \{y\})$ . Let  $z \in S_Z$  and let  $E$  be a finite-dimensional subspace of  $Z$  containing  $z$ . Choose  $T : E \rightarrow X^{**}$  satisfying (i), (ii) and (iv). We have  $(1 - \varepsilon) < (1 + \varepsilon)^{-1}$  so by (ii)

$$(1 - \varepsilon) \leq \|Tz\| \leq (1 + \varepsilon)$$

hence

$$|||Tz\| - 1| \leq \varepsilon$$

Using (iv) we get

$$\|z - 2\frac{Tz}{\|Tz\|}\| \leq \|z - 2Tz\| + 2\|Tz - \frac{Tz}{\|Tz\|}\| \leq (1 + \varepsilon) + 2|1 - \|Tz\|| \leq 1 + 3\varepsilon,$$

which shows that

$$\inf_{x \in S_X} \|z - 2x\| = 1.$$

By Theorem 2.4 in [LL09]  $X$  is a strict u-ideal in  $Z$ . This is true for any  $y \in Y$  and so, by Proposition 2.1 in [LL09],  $X$  is a strict u-ideal in  $Y$ .  $\square$

### 3. SUPER-IDEALS INHERIT DIAMETER 2 PROPERTIES AND THE DAUGAVET PROPERTY

Let  $X$  be a non-trivial (real) Banach space with unit ball  $B_X$ . By a slice of  $B_X$  we mean a set of the type  $S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}$  where  $x^*$  is in the unit sphere  $S_{X^*}$  of  $X^*$  and  $\varepsilon > 0$ . A finite convex combination of slices of  $B_X$  is then a set of the form

$$S = \sum_{i=1}^n \lambda_i S(x_i^*, \varepsilon_i), \quad \lambda_i \geq 0, \quad \sum_{i=1}^n \lambda_i = 1,$$

where  $x_i^* \in S_{X^*}$  and  $\varepsilon_i > 0$  for  $i = 1, 2, \dots, n$ .

The interrelation of the following three successively stronger properties were investigated in [ALN12]:

**Definition 3.1.** A Banach space  $X$  has the

- (i) *local diameter 2 property* if every slice of  $B_X$  has diameter 2.
- (ii) *diameter 2 property* if every non-empty relatively weakly open subset of  $B_X$  has diameter 2.
- (iii) *strong diameter 2 property* if every finite convex combination of slices of  $B_X$  has diameter 2.

It is not known to us whether properties (i) and (ii) really are different. Recently it has however been proved that property (iii) is indeed strictly stronger than (i) and (ii) [HLP]. The study of property (ii) above goes back to Shvidkoy's work [Shv00] on the Daugavet property, where a bi-product is that spaces with the Daugavet property enjoy the diameter 2 property, and to Nygaard and Werner's paper [NW01] where uniform algebras are shown to have the diameter 2 property. We postpone the definition of the Daugavet property till needed; it can be found in the introduction to Proposition 3.8 below. A uniform algebra is a separating closed subalgebra of a  $C(K)$ -space which contains the constants.

In addition to Daugavet spaces and uniform algebras, spaces with “big” centralizer is also known to have the diameter 2 property. Precise definition of “big” centralizer can be found in [ALN12] or [ABG10], we don't give it here as we will not really need it; for our purposes it is enough to know that Daugavet spaces, uniform algebras and spaces with “big” centralizer form three large classes of spaces with the diameter 2 property.

In [ALN12] it was shown that the first two classes also enjoy the strong diameter 2 property. The third class is so far not fully investigated concerning the strong diameter 2 property, but a subclass of this class is the class of M-embedded spaces (spaces which are M-ideals in their biduals, see the introduction to Example 1), and this class has the strong diameter 2 property.

Concerning stability, in [ALN12] it was proved that if  $X$  and  $Y$  both have the (local) diameter 2 property, then  $X \oplus_p Y$  has the (local) diameter 2

property,  $1 \leq p \leq \infty$ . Surprisingly this does not hold for the strong diameter 2 property unless  $p = 1, \infty$  ([HLP] and [ALN12, Theorem 2.7 (iii)] and the remark after that theorem).

It is also natural to ask for stability when forming tensor products. Here, using that  $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{L}(Y, X^*)$ , one gets almost directly that  $X \widehat{\otimes}_\pi Y$  has the local diameter 2 property if  $X$  (or  $Y$ ) has. For the class of spaces with “big” centralizer there are strong and positive results in the very recent paper [ABGRP11], but in general the situation is not clear at this moment.

We believe it is folklore among researchers working on the diameter 2 property that  $X$  inherits the diameter 2 property from its bidual  $X^{**}$ , although we don’t know any explicit reference for it. Here we will show the much more general result that all the diameter 2 properties passes on to super-ideals.

**Proposition 3.2.** *Let  $X$  be a super-ideal in a Banach space  $Y$ . If  $Y$  has the diameter 2 property, then so does  $X$ .*

*Proof.* Let  $U \subset B_X$  be relatively weakly open and  $\varepsilon > 0$ . It clearly suffices to prove that any set of the type

$$U_\delta = \{x \in B_X : |x_i^*(x - x_0)| < \delta, i = 1, 2, \dots, n, x_0 \in U\}$$

contains two points with distance  $> 2 - \varepsilon$ . In order to produce these two points, let

$$V_\delta = \{y \in B_Y : |\varphi x_i^*(y - x_0)| < \delta, i = 1, 2, \dots, n.\}$$

$V_\delta$  is relatively weakly open in  $B_Y$ , and so has diameter 2. Thus  $V_\delta$  contains two points  $y_1$  and  $y_2$  with distance  $> 2 - \varepsilon/4$ . By multiplying by some factor close to 1, we may assume that  $y_1$  and  $y_2$  both have norm strictly less than one, and with distance  $> 2 - \varepsilon/2$ .

Let  $E$  be spanned by  $x_0$  and the two points  $y_1, y_2$ , and let  $F = \text{span}\{x_i^*\}_{i=1}^n$ . Now, since  $X$  is a super-ideal in  $Y$ , we can find an “almost-isometric local projection”  $T$  such that  $Ty_1, Ty_2 \in B_X$  and  $\|Ty_1 - Ty_2\| > 2 - \varepsilon$ , and such that

$$|x_i^*(Ty_i - x_0)| = |x_i^*(Ty_i - Tx_0)| = |x_i^*(T(y_i - x_0))| = |\varphi x_i^*(y_i - x_0)| < \delta,$$

so  $Ty_1, Ty_2 \in U_\delta, i = 1, 2$ .  $\square$

*Remark 3.1.* Note that in the proof above we only needed to be able to push every three-dimensional  $E \subset Y$  into  $X$  almost isometrically.

Now we prove that also the strong diameter 2 property is inherited by super-ideals.

**Proposition 3.3.** *Let  $X$  be a super-ideal in Banach space  $Y$ . If  $Y$  has the strong diameter 2 property, then so does  $X$ .*

*Proof.* Let  $S \subset B_X$  be a finite convex combination of slices.  $S$  is then of the form

$$S = \sum_{i=1}^n \lambda_i S_i(x_i^*, \varepsilon_i),$$



where  $x_i^* \in B_X^*$ ,  $\varepsilon_i > 0$ ,  $\lambda_i > 0$ , and  $\sum_{i=1}^n \lambda_i = 1$ . Now put

$$S_\varphi = \sum_{i=1}^n \lambda_i S_{\varphi,i}(\varphi x_i^*, \varepsilon_i),$$

where  $\varphi$  is the Hahn-Banach extension operator associated with the super-ideal. Note that each  $S_{\varphi,i}(\varphi x_i^*, \varepsilon_i) = \{y \in B_Y : \varphi x_i^*(y) > 1 - \varepsilon_i\}$  is a slice of  $B_Y$ . Since  $S_\varphi$  has diameter 2, there are for every  $\eta > 0$ ,  $y_k \in S_\varphi$ ,  $k = 1, 2$ , such that  $\|y_1 - y_2\| > 2 - \eta$ . Now  $y_k \in S_\varphi$  is of the form  $y_k = \sum_{i=1}^{n_k} \lambda_i y_{\varphi,k}^i$  where  $y_{\varphi,k}^i \in S_{\varphi,i}(\varphi x_i^*, \varepsilon_i)$ . Let  $E = \text{span}(y_k, y_{\varphi,k}^{n_k})_{k,i} \subset Y$  and  $F = \text{span}(x_i^*)_i \subset X^*$ . By a perturbation argument, we can assume that  $\max_k \|y_k\| = r < 1$ .

For  $\delta > 0$  such that  $(1 + \delta) \cdot r \leq 1$ , choose  $T : E \rightarrow X$  which fulfills (i)-(iii) in the conclusion of Theorem 2.1 with this  $\delta$ , and observe that  $Ty_k = \sum_{i=1}^{n_k} \lambda_i T y_{\varphi,k}^i$ . Then  $Ty_k \in S$  since  $\|Ty_k\| \leq (1 + \delta)\|y_k\| \leq (1 + \delta) \cdot r \leq 1$ , and  $T y_{\varphi,k}^i(x_i^*) = \varphi x_i^*(y_{\varphi,k}^i) > 1 - \varepsilon_i$ , so  $T y_{\varphi,k}^i \in S_i$ .

Finally, observe that  $\|Ty_1 - Ty_2\| > (1 + \delta)^{-1}(2 - \eta)$  and that  $\delta$  and  $\eta$  can be chosen arbitrarily small, so the diameter of  $S$  must be 2.  $\square$

*Remark 3.2.* Note that in the proof above we can not take  $E$  with just 3 dimensions as we could in the proof of the similar result for the diameter 2 property.

**Corollary 3.4.** *Super-ideals inherit the local diameter 2 property.*

*Proof.* Take  $n = 1$  in the proof of Proposition 3.3.  $\square$

**Proposition 3.5.** *Let  $Y$  be a Banach space. If every infinite-dimensional separable ideal in  $Y$  has the (local, strong) diameter 2 property, then so does  $Y$ .*

*Proof.* First let us prove the result for the strong diameter 2 property. To this end let  $\varepsilon_i > 0$  for  $i = 1, \dots, n$  and  $S = \sum_{i=1}^k \lambda_i S_i$  a finite convex combination of slices  $S_i = \{y \in B_Y : y_i^*(y) > 1 - \varepsilon_i\}$  of the unit ball of  $Y$ . Each slice is relatively weakly open and therefore contains a ball of small radius about a point in the slice. Thus it is possible to find a sequence of infinitely many linearly independent points inside each slice. But then it is clear that this is also possible to do inside  $S$ . So let  $(y_n) \subset S$  be such a sequence, and let  $Z$  be the norm closure of  $\text{span}(y_n)$ . By [HM82] (cf. also [HWW93, Lemma III.4.3]) there is a separable ideal  $X$  in  $Y$  containing  $Z$  such that  $\text{span}(y_i^*)_{i=1}^k \subset \varphi(X^*)$  where  $\varphi : X^* \rightarrow Y^*$  is the Hahn-Banach extension operator. Now, for  $i = 1, \dots, k$ , find  $x_i^* \in X^*$  such that  $y_i^* = \varphi(x_i^*)$ . Let  $S'_i = \{x \in B_X : x_i^*(x) > 1 - \varepsilon_i\} = \{x \in B_X : \varphi x_i^*(x) > 1 - \varepsilon_i\}$  be slices of the unit ball of  $X$ . Denote by  $S' = \sum_{i=1}^k \lambda_i S'_i$  the corresponding convex combination of slices. Since  $S'$  has diameter 2 and  $S' \subset S$ ,  $S$  has diameter 2.

For the local diameter 2 property the result follows by taking  $k = 1$  in the argument above.

For the diameter 2 property let  $V$  be a relatively weakly open subset in  $B_Y$ . Find  $y_0 \in V$  and  $y_i^* \in Y^*$  such that  $V_\varepsilon = \{y \in B_Y : |y_i^*(y - y_0)| < \varepsilon, i = 1, \dots, n\} \subset V$ . It is possible to choose a sequence  $(y_n)$  of infinitely many

linearly independent points in  $V_\varepsilon$  and a similar argument as above will now finish the proof.  $\square$

Our next goal is to show that super-ideals inherit the Daugavet property. Let us first recall the definition of this property:

**Definition 3.6.** A Banach space  $X$  has the Daugavet property if, for every rank 1 operator  $T : X \rightarrow X$ ,

$$\|T + I\| = 1 + \|T\|.$$

In Definition 3.6,  $I$  denotes the identity operator on  $X$ . We will need a fundamental observation, from [KSSW00, Lemma 2.2]:

**Lemma 3.7.** *The following are equivalent.*

- (i)  $X$  has the Daugavet property.
- (ii) For all  $y \in S_X$ ,  $x^* \in S_{X^*}$  and  $\varepsilon > 0$  there exists  $x \in S_X$  such that  $x^*(x) \geq 1 - \varepsilon$  and  $\|x + y\| \geq 2 - \varepsilon$ .

The next result is proved for M-ideals in [KSSW00, Proposition 2.10].

**Proposition 3.8.** *If  $X$  is a super ideal in  $Y$  and  $Y$  has the Daugavet property, then  $X$  has the Daugavet property.*

*Proof.* Let  $\varphi : X^* \rightarrow Y^*$  be the (super) Hahn-Banach extension operator.

We will show that (ii) of Lemma 3.7 is fulfilled. For this, let  $y \in S_X$ ,  $x^* \in S_{X^*}$  and  $\varepsilon > 0$ . Consider the slice

$$S_1 = \{x \in B_X : x^*(x) \geq 1 - \varepsilon\}.$$

We will need to produce some  $x \in S_1$  with  $\|x\| = 1$  and  $\|y + x\| \geq 2 - \varepsilon$ . Look at

$$S = \{z \in B_Y : \varphi(x^*)(z) \geq 1 - \eta\}.$$

Since  $Y$  has the Daugavet property, for all  $\eta > 0$ , there is some  $z \in S$  with  $\|z\| = 1$  and such that  $\|z + y\| \geq 2 - \eta$ . Let  $\frac{\varepsilon}{2} > \eta > 0$  and choose  $\frac{\varepsilon}{2} > \delta > 0$  so small that  $\delta \leq \frac{\frac{\varepsilon}{2} - \eta}{2 - \frac{\varepsilon}{2}}$ . Note that this choice gives  $(1 + \delta)^{-1}(2 - \eta) \geq 2 - \frac{\varepsilon}{2}$ .

Let  $E = \text{span}\{z, y\} \subseteq Y$ ,  $F = \text{span}\{x^*\} \subseteq X^*$  and find a corresponding  $\delta$ -isometry  $T : E \rightarrow X$ . Let  $x = \frac{T(z)}{\|T(z)\|}$ . Clearly  $\|x\| = 1$ . We get

$$\|x - T(z)\| = |\|T(z)\| - 1| \leq \delta \leq \frac{\varepsilon}{2},$$

hence

$$\|x + y\| \geq \|T(z) + y\| - \|x - T(z)\| \geq (1 + \delta)^{-1}(2 - \eta) - \delta \geq 2 - \varepsilon.$$

Finally,

$$x^*(x) = x^*(T(z)) + x^*(x - T(z)) \geq \varphi(x^*)(z) - \delta \geq 1 - \eta - \delta \geq 1 - \varepsilon,$$

and we conclude that  $X$  has the Daugavet property.  $\square$

*Remark 3.3.* As in the proof of Proposition 3.2 the full strength of a super-ideal was not needed in the above proof. We only needed  $E$  2-dimensional and  $F$  1-dimensional. Of course, like for the diameter 2 properties, we also get from Proposition 3.8 that  $X$  inherits the Daugavet property from  $X^{**}$ , but this is trivial since, from the definition of the Daugavet property,  $X$  always has the Daugavet property if  $X^*$  has.

Let us observe that every Banach space can be realized as a 1-complemented subspace of a Banach space with the Daugavet property. In particular this implies that the converse of Proposition 3.3 does not hold.

**Proposition 3.9.** *Every Banach space  $X$  is a 1-complemented subspace of a space with the Daugavet property, hence also a 1-complemented subspace of a space with the strong diameter 2 property.*

*Proof.* It is a classical theorem of Milne that every Banach space is a 1-complemented subspace of a uniform algebra. Looking into its proof, as given in [GK01, Th 2.1], we see that this uniform algebra can be chosen to be the uniform algebra generated by  $X$  as a subspace in  $C(K)$ , where  $K$  is the dual unit ball equipped with the weak\* topology. Clearly  $K$  has no isolated points, and so this uniform algebra has the Daugavet property (see [Wer01, p. 79]). The last statement follows from [ALN12, Theorem 4.4].  $\square$

#### 4. GURARIĬ-SPACES IN TERMS OF SUPER-IDEALS

Recall that a Lindenstrauss space is a Banach space such that the dual is an  $L_1(\mu)$ -space for some (positive) measure  $\mu$ . Rao [Rao01, Proposition 1] has proved the following result.

**Theorem 4.1.** *For a Banach space  $X$  the following statements are equivalent:*

- (i)  $X$  is a Lindenstrauss space.
- (ii)  $X$  is an ideal in every superspace.

Below we will prove an analogous result for super-ideals. For this we will need the definition of a GurariĬ space.

**Definition 4.2.** A Banach space  $X$  is called a GurariĬ space if it has the property that whenever  $\varepsilon > 0$ ,  $E$  is a finite-dimensional Banach space,  $T_E : E \rightarrow X$  is isometric and  $F$  is a finite-dimensional Banach space with  $E \subset F$ , then there exists a linear operator  $T_F : F \rightarrow X$  such that

- (i)  $T_F(f) = T_E(f)$  for all  $f \in E$ , and
- (ii)  $(1 + \varepsilon)^{-1} \|f\| \leq \|T_F f\| \leq (1 + \varepsilon) \|f\|$  for all  $f \in F$ .

If  $T_F : F \rightarrow X$  can be taken to be isometric, then  $X$  is called a strong GurariĬ space.

GurariĬ proved in [Gur66] that GurariĬ spaces exist. Indeed he constructed a separable such Banach space and showed that all separable GurariĬ spaces are linearly almost isometric. Later Lusky [Lus76] proved that all separable GurariĬ spaces are indeed linearly isometric. The fact that strong GurariĬ spaces exist can be found in [GK11].

Let us now state and prove a similar result for super-ideals.

**Theorem 4.3.** *For a Banach space  $X$  the following statements are equivalent:*

- (i)  $X$  is a GurariĬ space.
- (ii)  $X$  is a super-ideal in every superspace.

*Proof.* (i)  $\Rightarrow$  (ii). Let  $X$  be a subspace of  $Y$ ,  $E$  a finite-dimensional subspace of  $Y$ , and  $\varepsilon > 0$ . If  $E \cap X$  is of dimension  $\geq 1$ , then let  $T : E \cap X \rightarrow X$  be the identity operator. By assumption there is a linear extension  $\hat{T}$  of  $T$  satisfying  $(1 + \varepsilon)^{-1}\|e\| \leq \|\hat{T}e\| \leq (1 + \varepsilon)\|e\|$  for every  $e \in E$ , just as needed. Now, if  $E \cap X = \{0\}$ , then choose some non-zero  $x \in X$ , put  $E' = \text{span}(E, \{x\})$  and argue as above.

(ii)  $\Rightarrow$  (i). Let  $\varepsilon > 0$  and choose  $\delta > 0$  such that  $(1 + \delta)^2 \leq 1 + \varepsilon$ . By [GK11, Theorem 3.6] we can assume  $X$  is a subspace of a Gurarii space  $X_G$ . Now, let  $E \subset F$  be finite-dimensional subspaces, and  $T : E \rightarrow X$  linear and isometric. Since  $X_G$  is a Gurarii space, there exists a linear extension  $\hat{T} : F \rightarrow X_G$  of  $T$  with  $(1 + \delta)^{-1}\|f\| \leq \|\hat{T}(f)\| \leq (1 + \delta)\|f\|$  for every  $f \in F$ . Put  $H = \hat{T}(F)$ . Since  $X$  is a super-ideal in  $X_G$ , there exists an operator  $S : H \rightarrow X$  satisfying  $(1 + \delta)^{-1}\|h\| \leq \|Sh\| \leq (1 + \delta)\|h\|$  for every  $h \in H$  such that  $Sh = h$  for every  $h \in H \cap X$ . It follows that  $S \circ \hat{T} : F \rightarrow X$  is a linear extension of  $T$  satisfying  $(1 + \varepsilon)^{-1}\|f\| \leq \|S \circ \hat{T}(f)\| \leq (1 + \varepsilon)\|f\|$  for every  $f \in F$ , which is exactly as desired.  $\square$

*Remark 4.1.* It follows from the techniques used in [GK11] that every non-separable Banach space can be isometrically embedded in a strong Gurarii space. Thus by arguing as in Theorem 4.3 it is easily seen that strong Gurarii spaces are exactly the spaces which are super-ideals with  $\varepsilon = 0$  in every superspace.

**Corollary 4.4.** *The separable Gurarii space is the only Banach space which is a super-ideal in every superspace.*

**Corollary 4.5.** *Gurarii spaces enjoy the Daugavet property and hence the strong diameter 2 property.*

*Proof.* Let  $X$  be a Gurarii space. Since  $X$  embeds isometrically into  $C(B_{X^*}, \text{weak}^*)$ , and  $C(B_{X^*}, \text{weak}^*)$  has the Daugavet property (cf. e.g. [Wer01]), the result follows Theorem 4.3 and Proposition 3.8.  $\square$

It is clear from Theorems 4.1 and 4.3 that a Gurarii space is a Lindenstrauss space. Thus, the last part of Corollary 4.5 is also a particular case of the following result.

**Proposition 4.6.** *Every infinite-dimensional Lindenstrauss space has the strong diameter 2 property.*

*Proof.* Let  $X$  be a Lindenstrauss space. It is classical that  $X^*$  is order isometric to an  $\ell_1$ -sum of  $L_1(\mu_a)$ -spaces where  $\mu_a$  is a probability measure (see e.g. [LT79, Theorem 1.b.2]).

Now there are two possibilities. Either every  $\mu_a$  is purely atomic, and then  $X^*$  is isometric to  $\ell_1(\Gamma)$  for some set  $\Gamma$ , or one  $\mu_a$  is not purely atomic (see [Lac74, Theorem 5.14.9] for a concrete representation). In the first case  $X^{**} = \ell_\infty(\Gamma)$  which has the strong diameter 2 property. In the latter case we may write  $X^{**} = Z \oplus_\infty L_\infty(\mu_a)$  and thus  $X^{**}$  has the strong diameter 2 property by Proposition 4.6 in [ALN12]. In either case  $X^{**}$  has the strong diameter 2 property, and by Proposition 3.3 so does  $X$ .  $\square$

Note that not all Lindenstrauss spaces, e.g.  $c_0$ , have the Daugavet property, see also [Wer01, p. 79].

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